NASA'S SATELLITE ORBIT ANOMALY PROBLEM CAN BE SOLVED PRECISELY IN THE FRAME OF EINSTEIN'S SPECIAL THEORY OF RELATIVITY. ANOMALY CONFIRMS THAT GRAVITY FIELDS PROPAGATE WITH VELOCITY OF LIGHT AS EINSTEIN PREDICTED

VICTOR M. BOGDAN

Abstract. NASA's Jet Propulsion Laboratory put on You Tube a problem that has been baffling the scientists for sometime. It involves an unexpected force acting on the space probes.

The author proves that NASA'S satellite orbit anomaly problem can be solved in the frame of Einstein's Special Theory of Relativity. The anomaly confirms that gravity fields propagate with velocity of light as Einstein predicted.

The proof is based on the authors discovery of the relativistic version of Newton's gravity field. The author provides formulas for relativistic equation of motion for a spacecraft in the joint gravitational field of the Earth and the Sun in a Lorentzian frame attached to the Earth. The formulas are suitable for digital computers and can be easily implemented. He also shows how to find solutions of the relativistic equations of motion for the spacecraft.

1. NASA is baffled by an unexpected force acting on space probe

The following is a transcript from You Tube video apparently made by Mr. Anderson from NASA's Jet Propulsion Laboratory [1]:

Mysteriously five space craft that flew past the earth have each displayed unexpected anomalies in their motions. **Pioneer anomaly** *has hints that unexpected forces may appear to act on spacecraft. The anomalies were seen with identical Pioneer 10 and 11 spacecraft. Both seem to experience tiny but unexplained constant acceleration toward the sun.*

In five of the six flybys, the scientists have confirmed anomalies. "I'm feeling both humbled and perplexed" said Anderson. "There is something strange going on with spacecraft, with respect to Earth's equator. It suggests that the anomaly is related to Earth's rotation"

"Another thing in common between the Pioneer and these flybys is what you would call an **unbound orbit around a central body,"** *Anderson said. "For*

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instance, the Pioneers are flying out of the system. There's definitely something going on."

"Could Earth's rotation be a part of the equation? Rotation forecasts are important to NASA's Jet Propulsion Laboratory."

"The more forceful winds double the angular momentum of the atmosphere. Winds can effect the Earth's rotation."

"Something's effecting the satellites orbit. Could it be Planet X? What do you think?"

The purpose of this note is to answer the questions rased in this video and to provide the gravity field formulas in the frame of Einstein's special theory of relativity. These formulas can be used for navigation in gravitational fields in the presence of several celestial bodies in motion.

2. The solution of the problem

The answer comes from Einstein's special theory of relativity involving Maxwell's equations and formulas discovered by the author for relativistic gravity field that correct the Newton's formula. The anomaly indeed is due to Earth's rotation.

With respect to the Lorentzian frame attached to Earth the Sun moves around the Earth almost periodically with period of 24 hours. The distance to the Sun is approximately

$$r = 8\,[min] \times c\,[km/sec]$$

so the circumference of Sun's orbit around the Earth in the Lorentzian frame is $2\pi r\cos(\alpha)$, where α is the angle between the plane of Earth's equator and the plane of Sun's orbit, and the speed of the Sun in that frame is approximately $\frac{1}{30}\cos(\alpha)\,c$.

This fraction

$$q = \frac{1}{30}\cos(\alpha)$$

represents an important factor. It determines the rate of convergence in the computation of the field representing the retarded time.

For the sake of simplicity select the unit of time so that the speed of light in the Lorentzian frame is $c = 1$. Select also the unit of mass so that gravitational constant $G = 1$.

This means that if the unit of length is 1 meter, then the unit of time is approximately 3.3 nanoseconds and the unit of rest mass is about 3.8 metric tons.

Assume that we have the formula $t \mapsto r(t) \in R^3$ for the trajectory of the Sun around the Earth in our Lorentzian frame. It is almost periodic. So we may a priori estimate a bound on its velocity $w(t) = \dot{r}(t)$ in that frame. In our case it is q.

Let the pair $(y, t) \in R^3 \times R$ denote any point in the Lorentzian frame. According to Einstein's theory of relativity [9] and his work jointly with Rosen [10] on gravity, gravity waves propagate through space with velocity of light.

For a gravity wave from the Sun to arrive at a point $y \in R^3$ at time t, the wave should be emitted from position $r(\tau)$ on its trajectory at some earlier time τ. This leads to the Lorentz relation

$$t - \tau = |y - r(\tau)|$$

Since the Sun's trajectory is fixed the above equality implicitly defines the function $(y,t) \mapsto \tau$. Notice that for given (y,t) the value τ is a fixed point of the function $f(s) = t - |y - r(s)|$ considered on the interval $s \leq t$. The function f is a contraction and its Lipschitz constant is q. From Banach's fixed-point theorem, we get a fast converging algorithm for computing the function $\tau(y,t)$. Namely use the recursive formula

$$\tau_0(y,t) = 0 \quad \text{and} \quad \tau_n(y,t) = f(\tau_{n-1}(y,t) \quad \text{for all} \quad (y,t),\ n = 1,2,\ldots$$

After n steps we get the following approximation

$$|\tau(y,t) - \tau_n(y,t)| \leq \frac{q^n}{1-q} |t - |y - r(0)|\,| \quad \text{for all} \quad (y,t) \in R^4,\ n = 1,2,\ldots$$

The function τ is continuous as follows from results of Bogdan [5], Theorem 3.1. Introduce the scalar field T called the time delay field by the formula

$$T(y,t) = t - \tau(y,t) \quad \text{for all} \quad (y,t) \in R^4$$

and vector field

$$r_{12}(y,t) = y - r(\tau(y,t)) \quad \text{for all} \quad (y,t) \in R^4$$

representing a vector starting on the trajectory of the Sun at retarded time and having its end at the point $y \in R^3$ at time t. It replaces Newton's radius vector. Notice the equality

$$T(y,t) = |r_{12}(y,t)|$$

for all $(y,t) \in R^4$.

Introduce also the velocity v and acceleration a vector fields by the formulas

$$v(y,t) = w(\tau(y,t))$$
$$a(y,t) = \dot{w}(\tau(y,t))$$

for all $(y,t) \in R^4$.

On the set

$$G = \left\{(y,t) \in R^4 :\ T(y,t) > 0\right\}$$

introduce the scalar field $u = 1/T$, the unit vector field $e = ur_{12}$, representing direction of vector r_{12}, and the scalar field

$$z = \frac{1}{1 - e \cdot v}.$$

Notice that since the velocity field $|v| \leq q < c = 1$ we have for the dot product

$$|e \cdot v| \leq q$$

so the field z is well defined on the set G and we have the uniform estimate

$$0 \leq z \leq \frac{1}{1-q}.$$

Definition 2.1 (Fundamental fields)**.** *The fields*

$$\tau,\ r_{12},\ v,\ a,\ T,\ u,\ z,\ e$$

will be called the **fundamental fields** *associated with the trajectory of the moving point mass.*

The fundamental fields are continuous on their respective domains. This follows from the fact that composition of continuous functions yields a continuous function. Thus all of them, for sure, are continuous on the open set G of points that do not lie on the trajectory.

We would like to stress here that the fundamental fields depend on the Lorentzian frame, in which we consider the trajectory. It is important to find expressions involving fundamental fields that yield fields invariant under Lorentzian transformations.

Lorentz [12] and, independently, Einstein [9], Part II, section 6, established that fields satisfying Maxwell equations are invariant under Lorentzian transformations.

Now consider the fields E, B defined by means of fundamental fields by the formulas

$$\begin{aligned} E &= u^2 e + u^{-1} D(u^2 e) + D^2(e) \\ B &= e \times E \end{aligned}$$

where D denotes the partial derivative with respect to time $\frac{\partial}{\partial t}$.

The pair (E, B) of fields associated with a trajectory of a moving point mass in a given Lorentzian frame satisfies homogenous Maxwell equations. Therefore according to a result of Lorentz and Einstein, the pair of fields (E, B) is invariant under Lorentzian transformations. The fields E and B, obtained purely by mathematical construction, form parts of an antisymmetric tensor of second rank.

Indeed, the pair of fields

$$E = (E_1, E_2, E_3) \quad \text{and} \quad B = (B_1, B_2, B_3)$$

is a part of an antisymmetric tensor that in Lorentzian space-time $R^3 \times R$ frame has the matrix that looks as follows

$$\begin{bmatrix} 0 & +E_1 & +E_2 & +E_3 \\ -E_1 & 0 & +B_3 & -B_2 \\ -E_2 & -B_3 & 0 & +B_1 \\ -E_3 & +B_2 & -B_1 & 0 \end{bmatrix}$$

When the point mass is in its rest frame the field E differs from Newton's gravity field just by a constant of proportionality. The formula for the field E amends the formula derived heuristically by Feynman for a moving point charge. So it is proper to call the field E the **Newton-Feynman field** generated by the trajectory.

The following theorem permits us to analyze such fields E and B by reducing the computations to simple algebraic expressions involving the fundamental fields.

Introduce operators $D = \frac{\partial}{\partial t}$ and $D_i = \frac{\partial}{\partial x_i}$ for $i = 1, 2, 3$ and $\nabla = (D_1, D_2, D_3)$. Observe that δ_i in the following formulas denotes the i-th unit vector of the standard base in R^3 that is $\delta_1 = (1, 0, 0)$, $\delta_2 = (0, 1, 0)$, $\delta_3 = (0, 0, 1)$.

Theorem 2.2 (Partial derivatives of fundamental fields)**.** *Assume that in some Lorentzian frame we are given an admissible trajectory $t \mapsto r_2(t)$. Define the time derivative $w(t) = \dot{r}_2(t)$. For partial derivatives with respect to coordinates of the*

vector $r_1 = y \in R^3$ we have the following identities on the set G

$$D_iT = ze_i \quad \textit{where} \quad v = \dot{r}_2 \circ \tau, \tag{2.1}$$
$$D_iu = -zu^2e_i, \tag{2.2}$$
$$D_iv = -e_iza \quad \textit{where} \quad a = \dot{w} \circ \tau, \tag{2.3}$$
$$D_i\tau = -ze_i, \tag{2.4}$$
$$D_ie = -uze_ie + u\delta_i + uze_iv \quad \textit{where} \quad \delta_i = (\delta_{ij}), \tag{2.5}$$
$$D_iz = -z^3e_i\langle e, a\rangle - uz^3e_i + uz^2e_i + uz^2v_i + uz^3e_i\langle v, v\rangle \tag{2.6}$$

and for the partial derivative with respect to time we have

$$DT = 1 - z, \tag{2.7}$$
$$Du = zu^2 - u^2, \tag{2.8}$$
$$D\tau = z, \tag{2.9}$$
$$Dv = za \quad \textit{where} \quad a = \dot{w} \circ \tau, \tag{2.10}$$
$$De = -ue + uze - uzv, \tag{2.11}$$
$$Dz = uz - 2uz^2 + z^3\langle e, a\rangle + uz^3 - uz^3\langle v, v\rangle. \tag{2.12}$$

Since the expression on the right side of each formula represents a continuous function, the fundamental fields are at least of class C^1 on the set G. Moreover if the trajectory is of class C^∞ then also the fundamental fields are of class C^∞ on G.

3. Bogdan-Feynman Theorem for a moving point mass

The following theorem is just a consequence that the wave emitted from a trajectory of a point mass propagates in a Lorentzian frame with the speed of light. Please notice that the physical nature of the fields is completely irrelevant.

The important fact is that we are working in a Lorentzian frame, in which we are given a trajectory, on which the velocity and acceleration are bounded. Such trajectory alone generates in a unique way the system of the fundamental fields by means of which we are able to construct fields that are preserved by Lorentzian transformations.

We remind the reader that we are working in a fixed Lorentzian frame. The position vector is denoted by $r_1 \in R^3$ and time by $t \in R$. The position axes are oriented so as to form right screw orientation. The units are selected so that the speed of light $c = 1$.

Partial derivatives with respect to coordinates of r_1 are denoted by D_1, D_2, D_3 and with respect to time just by D. The gradient differential operator is denoted by $\nabla = (D_1, D_2, D_3)$ and the D'Alembertian operator by $\Box^2 = \nabla^2 - D^2$. The expression $\langle e, v\rangle$ denotes the dot product in R^3 of the vectors involved.

Theorem 3.1 (Bogdan-Feynman Theorem)**.** *Assume that in a given Lorentzian frame the map $t \mapsto r_2(t)$ from R to R^3 represents an admissible trajectory of class C^3. Assume that G denotes the open set of points that do not lie on the trajectory. All the following field equations are satisfied on the entire set G.*

Consider the pair of fields E and B over the set G given by the formulas

$$E = u^2e + u^{-1}D(u^2e) + D^2e \quad \textit{and} \quad B = e \times E$$

where u and e represent fundamental fields (2.1) associated with the trajectory $r_2(t)$.

Then this pair of fields will satisfy the following homogenous system of Maxwell equations

$$\nabla \times E = -DB, \quad \nabla \cdot E = 0,$$
$$\nabla \times B = +DE, \quad \nabla \cdot B = 0,$$

and the homogenous wave equations

$$\Box^2 E = 0, \qquad \Box^2 B = 0.$$

Moreover Liénard-Wiechert potentials, expressed in terms of the fundamental fields as $A = uzv$ and $\phi = uz$, satisfy the homogenous system of wave equations with Lorentz gauge formula

$$\Box^2 A = 0, \quad \Box^2 \phi = 0, \quad \nabla \cdot A + D\phi = 0$$

and generate the fields E and B by the formulas

$$E = -\nabla\phi - DA \quad \text{and} \quad B = \nabla \times A.$$

Finally we have the following explicit formula for the field E in terms of the fundamental fields

$$\begin{aligned} E = &-uz^2 a + uz^3\langle e, a\rangle e - uz^3\langle e, a\rangle v \\ &+ u^2z^3 e - u^2z^3\langle v, v\rangle e - u^2z^3 v + u^2z^3\langle v, v\rangle v. \end{aligned}$$

For a proof of the above theorem see Bogdan [5], Theorem 11.1. As a consequence of the above theorem the components of the quantities E, B, A, and ϕ, since they satisfy the homogenous wave equation $\Box^2 y = 0$, propagate in the Lorentzian frame with velocity of light $c = 1$.

4. Relativistic equation of motion for a spacecraft in the joint gravitational field of the Earth and the Sun in a Lorentzian frame attached to the Earth

Now for a body of rest mass m_0 moving under the influence of a force F from Newton-Einstein formula, time derivative of the momentum is equal to force, we must have

$$\dot{p} = F$$

where $p = m_0\gamma\, v$ is the relativistic momentum and $\gamma(v) = (1-|v|^2)^{-1/2}$. Computing the derivative of p with respect to time we get

$$\begin{aligned} \dot{p} &= m_0\,\gamma\,\dot{v} + m_0\,\dot{\gamma}\,v = m_0\,\gamma\,\dot{v} + m_0\,\gamma^3\ \langle v, \dot{v}\rangle\ v \\ &= m_0\gamma\,(\dot{v} + \gamma^2\ \langle v, \dot{v}\rangle\ v) = m_0\,\gamma(v)\,\Gamma(v)(\dot{v}), \end{aligned}$$

where $\Gamma(v)$ is the linear transformation of R^3 into R^3 given by the formula

$$\Gamma(v)(h) = h + \gamma^2\,\langle v, h\rangle\, v \quad \text{for all} \quad h \in R^3. \tag{4.1}$$

Notice that for fixed velocity v the transformation $\Gamma(v)$ represents a symmetric, positive definite transformation with eigenvalues equal respectively to

$$(1 + \gamma^2|v|^2),\ 1,\ 1.$$

Thus the inverse transformation $\hat{\Gamma}(v)$ exists and is also symmetric. Since its eigenvalues are

$$(1+\gamma^2|v|^2)^{-1},\ 1,\ 1$$

and the norm of positive definite symmetric transformation is its maximal eigenvalue, we must have

$$|\hat{\Gamma}(v)| = 1 \quad \text{for all} \quad v \in R^3$$

and we have also

$$\hat{\Gamma}(v)\Gamma(v) = e \quad \text{for all} \quad v \in R^3$$

where e denotes here the identity transformation in the algebra $Lin(R^3, R^3)$ that is $e(h) = h$ for all $h \in R^3$.

We shall call the function $v \mapsto \hat{\Gamma}(v)$ the **reciprocal to the function** $\Gamma(v)$, since

$$\hat{\Gamma}(v) = [\Gamma(v)]^{-1} \quad \text{for all} \quad v \in R^3.$$

5. The mystery force identified

Thus the relativistic equation of motion of the spacecraft can be written as

$$m_0\,\gamma(v)\Gamma(v)(\dot{v}) = -m_e\,E_e - m_s\,E_s \tag{5.1}$$

where m_e and m_s denote the rest mass of the Earth and of the Sun, and where E_e and E_s represent the Newton-Feynman fields generated by the Earth and the Sun respectively. The **mystery force** is $F_s = -m_s\,E_s$ and as Mr. Anderson suspected it is due to Earth's rotation, or equivalently due to the fact that in a Lorentzian frame attached to Earth the Sun moves around the Earth.

6. Algorithm for computing the transformation Gamma hat

The space $Lin(U, U)$ of linear bounded operators from a Banach space U into itself beside being closed under addition and scalar multiplication is also closed under the composition of operators: $P \circ Q \in Lin(U, U)$ for all $P, Q \in Lin(U, U)$. This operation has the property that $|P \circ Q| \le |P|\,|Q|$.

Definition 6.1 (Banach Algebras)**.** *A Banach space U is called a* **Banach algebra,** *if it is equipped with a bilinear operation $(u, w) \to u\,w$, from the product $U \times U$ into U, that is associative: $(u\,w)\,z = u\,(w\,z)$, and such that*

$$|u\,w| \le |u|\,|w| \quad \textit{for all} \quad u,\, w \in U.$$

If in addition there is an element e in U such that $e\,u = u\,e = u$ for all $u \in U$, then such an algebra is called a **Banach algebra with unit.** *In such a case an element u is called invertible if for some $w \in U$, called the inverse of u, we have*

$$u\,w = w\,u = e.$$

The unit element and the inverse elements are unique. We denote the inverse of u by u^{-1}.

The following proposition is very useful in numerical computations of the inverse transformations. It represents a simple application of the Banach fixed point theorem and it is essential in the development of the theory presented in this note. It represents Pro. 6.25 of Bogdan [4].

Proposition 6.2 (Inverse $(e-v)^{-1}$ exist if $|v|<1$)**.** *Let U be a Banach algebra with unit. Then for every element $v \in U$ such that $|v|<1$ the inverse $w=(e-v)^{-1}$ exists. It is a fixed point of the operator f defined by the following formula*

$$f(u)=e+vu \quad \text{for all} \quad u \in U.$$

Moreover the sequence $w_n=f(w_{n-1})$ where $w_0=0$ is of the form

$$w_n=e+v+v^2+\cdots+v^n=\sum\nolimits_{0\le k\le n} v^k \quad \text{for all} \quad n>0$$

and we have the following estimate for the distance of the fixed point w and the approximation w_n

$$|w-w_n| \le \frac{|v|^n}{1-|v|} \quad \text{for all} \quad n>0,$$

and we also have an explicit formula for the inverse element w as the sum of an absolutely convergent series

$$w=e+v+v^2+\cdots=\sum\nolimits_{n\ge 0} v^n.$$

7. Returning to our problem at hand

Notice first of all that the space $Lin(R^3,R^3)$ of all linear transformations from R^3 into itself can be identified with the space of 3×3 matrices. If e denotes the unit matrix let $H(v)$ denote the transformation defined by the formula

$$H(v)(h)=-\gamma(v)^2\ \langle v,h\rangle\ v \quad \text{for all} \quad h \in R^3.$$

The norm of this transformation is $\frac{|v|^2}{1-|v|^2}$. So if the velocity $|v|$ stays below $\sqrt{0.5}=0.707\ldots$ that is below the 70% of the speed of light the norm of the operator H will be less then 1. So we can use the above proposition to compute the inverse transformation

$$\hat{\Gamma}(v)=(e-H(v))^{-1}$$

with any level of precision.

8. Algorithm for computing relativistic trajectory of spacecraft

So here we are. We can rewrite the equation of motion of the spacecraft (5.1) in the equivalent form as

$$\begin{aligned}\dot{v}&=(m_0\gamma(v))^{-1}\hat{\Gamma}(v)(-m_e\,E_e-m_s\,E_s)=f(y,t,v),\\ \dot{y}&=v,\end{aligned}$$

and this system of differential equations we can solve, say, by means of Runge-Kutta methods with any level of precision.

In closing I would like to thank Mr. Anderson from Jet Propulsion Laboratory for putting this problem onto You Tube.

Taking this opportunity I would like to thank also to my former colleagues, Donald Jezewski and Victor Bond, from Mission Planning and Analysis Division of NASA's Lyndon B. Johnson Space Center in Houston, Texas, for introducing me to problems of celestial mechanics.

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Department of Mathematics, McMahon Hall 207, CUA, Washington DC 20064, USA
E-mail address: bogdan@cua.edu